The residue theorem from a numerical perspective

Robin K. S. Hankin

Abstract

A short vignette illustrating Cauchy's integral theorem using numerical integration

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In this very short vignette, I will use contour integration to evaluate

$$\int_{x=-\infty}^{\infty} \frac{e^{ix}}{1+x^2}\,dx \tag{1}$$

using numerical methods. This document is part of the **elliptic** package (Hankin 2006).

The residue theorem tells us that the integral of $f(z)$ along any closed nonintersecting path, traversed anticlockwise, is equal to $2\pi i$ times the sum of the residues inside it.

Take a semicircular path P from $-R$ to $+R$ along the real axis, then following a semicircle in the upper half plane, of radius R to close the loop (figure 1. Now consider large R. Then P encloses a pole at i [there is one at $-i$ also, but this is outside P, so irrelevent here] at which the residue is $-i/2e$. Thus

$$\oint_P f(z)\,dz = 2\pi i \cdot (-i/2e) = \pi/e \tag{2}$$

along P; the contribution from the semicircle tends to zero as $R \longrightarrow \infty$; thus the integral along the real axis is the whole path integral, or π/e.

We can now reproduce this result analytically. First, choose R:

```
> R <- 400
```

And now define a path P. First, the semicircle:

```
> u1     <- function(x){R*exp(pi*1i*x)}
> u1dash <- function(x){R*pi*1i*exp(pi*1i*x)}
```

and now the straight part along the real axis:

```
> u2     <- function(x){R*(2*x-1)}
> u2dash <- function(x){R*2}
```

And define the function:

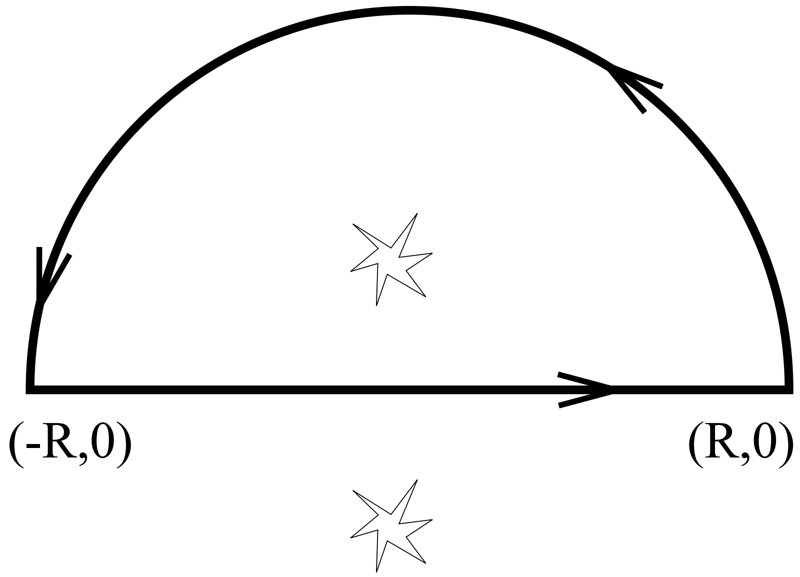

Figure 1: Contour integration path from $(-R, 0)$ to $(R, 0)$ along the real axis, followed by a semicircular return path in the positive imaginary half-plane. Poles of $e^{ix}/(1 + x + 2)$ symbolised by explosions

```
> f <- function(z){exp(1i*z)/(1+z^2)}
```

Now carry out the path integral. I'll do it explicitly, but note that the contribution from the first integral should be small:

```
> answer.approximate <-
+     integrate.contour(f,u1,u1dash) +
+     integrate.contour(f,u2,u2dash)
```

And compare with the analytical value:

```
> answer.exact <- pi/exp(1)
> abs(answer.approximate - answer.exact)

[1] 6.244969e-07
```

Now try the same thing but integrating over a triangle instead of a semicircle, using `integrate.segments()`. Use a path P' with base from $-R$ to $+R$ along the real axis, closed by two straight segments, one from $+R$ to iR, the other from iR to $-R$:

```
> abs(integrate.segments(f,c(-R,R,1i*R))- answer.exact)

[1] 5.157772e-07
```

Observe how much better one can do by integrating over a big square instead:

```
> abs(integrate.segments(f,c(-R,R,R+1i*R, -R+1i*R))- answer.exact)

[1] 2.319341e-08
```

The residue theorem for function evaluation

If $f(\cdot)$ is holomorphic within C, Cauchy's residue theorem states that

$$\oint_C \frac{f(z)}{z-z_0} = f(z_0). \tag{3}$$

Function `residue()` is a wrapper that takes a function $f(z)$ and integrates $f(z)/\left(z-z_0\right)$ around a closed loop which encloses z_0. We can test this numerically:

```
> f <- function(z){sin(z)}
> numerical <- residue(f,z0=1,r=1)
> exact <- sin(1)
> abs(numerical-exact)

[1] 1.111203e-16
```

which is unreasonably accurate, IMO.

References

Hankin RKS (2006). "Introducing elliptic, an R package for elliptic and modular functions." *Journal of Statistical Software*, **15**(7).

Affiliation:

Robin K. S. Hankin
Auckland University of Technology
2-14 Wakefield Street
Auckland
New Zealand
E-mail: hankin.robin@gmail.com